Algebraic Geometry 1

Exam Solution

Problem 1:

1) Assume that for every $F \in I(X)$ we have $F(y)=0$.

It means $y \in V(I(X)) = \overline{X} = X$ Contradiction.

2) Let $X = X_1 \cup \dots \cup X_r$, X_i irreducible components.

Claim: X_i = one point.

WLOG assume that X_1 contains two distinct points $x, y \in X_i$.

X is Hausdorff $\Rightarrow \exists\, \mathcal{U}, V$ open in X, such that $x \in \mathcal{U}, y \in V$ and $\mathcal{U} \cap V = \emptyset$

Set $\mathcal{U}^c := X \setminus \mathcal{U}$, $V^c := X \setminus V$

Then $\mathcal{U}^c, V^c$ are closed and $y \in \mathcal{U}^c$, $x \in V^c$ (*)
$x \notin \mathcal{U}^c$, $y \notin V^c$

Since $\mathcal{U} \cap V = \emptyset \Rightarrow \mathcal{U}^c \cup V^c = X$

Then $X_1 = X_1 \cap X = X_1 \cap (\mathcal{U}^c \cup V^c) = (X_1 \cap \mathcal{U}^c) \cup (X_1 \cap V^c)$

By (*) $X_1 \cap \mathcal{U}^c$ and $X_1 \cap V^c$ are closed proper subsets of X_1.

We get a contradiction, since X_1 is irreducible.

It follows that any irreducible component X_i is a point.

Hence, $X = X_1 \cup \dots \cup X_r$ is a ~~union~~ finite union of points.

Problem 2

1) Clearly $L \subseteq V(X-Y, Y-Z)$

Now if $(x,y,z) \in V(X-Y, Y-Z)$ we get $x-y=0$ and $y-z=0$

$\Rightarrow x=y=z \Rightarrow (x,y,z) \in L. \Rightarrow V(X-Y,Y-Z) \subseteq L$

2) $I(L) = \{ F \in K[X,Y,Z] \mid F(t,t,t)=0 \ \forall\, t \in K \}$

$\operatorname{Ker}\theta = \{ F \in K[X,Y,Z] \mid \underbrace{F(T,T,T)}_{\text{as a polynomial in } K[T]}^{\theta(F)} = 0 \}$

Since (a polynomial $g(T) \in K[T]$ is zero $\Leftrightarrow$ $g(t)=0$ for every $t \in K$)

we get $I(L) = \operatorname{Ker}\theta$.

3) $X-Y$, $Y-Z$ are zero on L

$\Rightarrow (X-Y, Y-Z) \subseteq I(L) \underset{\text{by (2)}}{=} \operatorname{Ker}\theta$

~~By 2)~~ Hence, we have a K-alg. hom.

$$K[X,Y,Z]/(X-Y,Y-Z) \xrightarrow{\tilde{\theta}} K[T]$$

$\overline{F(X,Y,Z)} = F(\bar{X},\bar{Y},\bar{Z}) \longmapsto F(T,T,T)$, $\bar{X},\bar{Y},\bar{Z}$ classes of X,Y,Z

Note that

$(X-Y,Y-Z) = I(L) \Leftrightarrow \tilde{\theta}$ is injective. It is enough to prove the injectivity of $\tilde{\theta}$.

~~In~~ In $K[X,Y,Z]/(X-Y,Y-Z)$ we have $\bar{X}=\bar{Y}=\bar{Z}$ $\boxed{\text{for the classes of } X,Y,Z}$

~~Assume $\tilde{\theta}(\overline{F(X,Y,Z)})=$~~ Hence, $F(\bar{X},\bar{Y},\bar{Z}) = F(\bar{X},\bar{X},\bar{X})$

~~If $\theta($~~ Assume $\tilde{\theta}(\overline{F(X,Y,Z)}) = F(T,T,T) = 0$

Then $\overline{F(X,Y,Z)} = F(\bar{X},\bar{Y},\bar{Z}) = F(\bar{X},\bar{X},\bar{X}) = \overline{F(X,X,X)} = 0$

Hence, $\tilde{\theta}$ is injective.

Another way to show $(X-Y,Y-Z) = I(L)$: Using Euclidian division write any $F(X,Y,Z) \in K[X,Y,Z]$ as $F(X,Y,Z) = P_1(X,Y,Z)(X-Y) + R_1(Y,Z) =$
$= P_1(X,Y,Z)(X-Y) + P_2(Y,Z)(Y-Z) + R_2(Z)$. If $F(T,T,T)=0$, then $R_2(T)=0 \Rightarrow R_2=0$
$\Rightarrow F \in (X-Y,Y-Z)$

4)
$$\begin{aligned} \mathbb{A}^1 &\longrightarrow L \subseteq \mathbb{A}^3 \\ t &\longmapsto (t,t,t) \\ x &\longleftarrow (x,y,z) \end{aligned}$$

Clearly both maps are regular

inverse map.

$\dim L = \dim \mathbb{A}^1 = 1$

5) Let $a \in \mathcal{U}$. Then $\exists$ a neighborhood W_a of a, s.t. $f|_{W_a} = \frac{F}{G}|_{W_a}$ for some $F, G \in K[X_1, X_2, X_3]$ and G with no zeros on W_a.
We can assume F, G to be coprime.

Claim 1: $f = \frac{F}{G}$ on $\mathcal{U}$. (we did this in the lecture several times)

Let $b \in \mathcal{U}$ and $f = \frac{\tilde{F}}{\tilde{G}}$ on some neighborhood W_b of b. ($\tilde{G}$ has no zeros on W_b)

Then $\frac{F}{G} = \frac{\tilde{F}}{\tilde{G}}$ on $W_a \cap W_b \Rightarrow F\tilde{G} = \tilde{F}G$ on $W_a \cap W_b \leftarrow$ dense in $\mathbb{A}^3$

$\Rightarrow F\tilde{G} = \tilde{F}G$ as polynomials $\xRightarrow[F,G \text{ coprime}]{} G \mid \tilde{G} \Rightarrow \tilde{G} = HG$ and $\tilde{F} = HF$ for some $F \in K[X_1, X_2, X_3]$

Hence, $\frac{\tilde{F}}{\tilde{G}} = \frac{FH}{GH} = \frac{F}{G}$ on $W_b \Rightarrow$

$\Rightarrow f = \frac{F}{G}|_{\mathcal{U}}$ and moreover G has no zeros on $\mathcal{U}$.

Claim 2: $G = \text{const}$ (that is $\deg G = 0$)

If $G \neq \text{const}$, then $\dim V(G) = \dim \mathbb{A}^3 - 1 = 2$

It follows $V(G) \cap \mathcal{U} \neq \emptyset$ (otherwise $\underset{\dim = 2}{V(G)} \subset \underset{\dim = 1}{L}$ $\lightning$)

But on the other hand, G has no zeros on $\mathcal{U}$ by Claim 1. Hence, $G = \text{const}$ and $f = F|_{\mathcal{U}}$.

6) Recall from the lecture that $A(\mathbb{A}^3) = K[X, Y, Z]$.
So the surjectivity of i^* follows from 5).
Note that $\mathbb{A}^3$ is irreducible, hence, $\mathcal{U}$ is dense in $\mathbb{A}^3$.
If for two polynomials $F, G \in K[X, Y, Z]$, we have $F|_{\mathcal{U}} = G|_{\mathcal{U}}$,
then $\mathcal{U} \subset \underbrace{V(F-G)}_{\text{closed}} \Rightarrow V(F-G) = \mathbb{A}^3 \Rightarrow F = G \Rightarrow i^*$ is injective.

7) ~~Since φ~~ We use the hint. Since $\varphi: V \to \mathcal{U}$ is an iso, $f \searrow \mathbb{A}^1$
then $\varphi^*: A(\mathcal{U}) \to A(V)$ is also an isomorphism.
$f \longrightarrow f \circ \varphi$

Then for the composition $i \circ \varphi: V \xrightarrow{\varphi} U \xrightarrow{i} \mathbb{A}^3$
we have $(i \circ \varphi)^* = \varphi^* \circ i^*: A(\mathbb{A}^3) \to A(V)$ is an iso of K-algebras,
since both φ^* and i^* are isomorphisms (i^* by question 6)
Since $\mathbb{A}^3$ and V are affine alg. sets it follows from the lecture that $i \circ \varphi: V \to \mathbb{A}^3$

is an isomorphism, but it is not surjective.

Problem 3

1) $x_1x_2,\ x_0x_2,\ x_0x_1$ hom. polynomials of the same degree $=2$ and do not have a common zero on $\mathcal{U}$.

2) Consider $\varphi\circ\varphi : \mathcal{U} \to \mathcal{U}$

$$\varphi\circ\varphi([x_0:x_1:x_2]) = \varphi([x_1x_2 : x_0x_2 : x_0x_1]) =$$

$$= [x_0^2x_1x_2 : x_0x_1^2x_2 : x_0x_1x_2^2] \underset{\substack{x_0x_1x_2 \neq 0 \\ \text{so we can divide} \\ \text{all coordinates} \\ \text{by } x_0x_1x_2}}{=\!=} [x_0 : x_1 : x_2]$$

Hence, $\varphi^2 = id_{|\mathcal{U}} \Rightarrow \varphi$ is an isomorphism $(\varphi^{-1} = \varphi)$.

Another solution: Since $U \subseteq \mathcal{U}_0 \simeq \mathbb{A}^2$, we get $\varphi : \overset{\subseteq \mathbb{A}^2}{\mathcal{U}} \longrightarrow \mathcal{U} \subseteq \mathbb{A}^2$, $(x,y) \mapsto \left(\frac{1}{x}, \frac{1}{y}\right)$ ← check this!, which is clearly an isomorphism $(\varphi^2 = id)$.

Problem 4 1) Choose 3 different points $v, u, w \in X \subseteq \mathbb{P}^3$

Then there exists a hyperplane $H \subseteq \mathbb{P}^3$ containing v, u, w.

(Indeed, the corresponding vectors $\vec{v}, \vec{u}, \vec{w} \in \mathbb{A}^4$ lie in some 3-dim K-vector subspace $U \subseteq K^4$, which is of codim $= 1$ in K^4.

Hence U is given by one linear equation $f = 0$, $f \in K[x_0, x_1, x_2, x_3]$ and we get $v, u, w \in H = V(f) \subseteq \mathbb{P}^3$.)

By Bézout's Theorem, if $X \not\subset V(f)^{=H}$, then

$$3 \underset{\uparrow}{\leq} |X \cap H| \leq \deg X \cdot \deg f = \cancel{\deg}\ 2 \cdot 1 = 2$$

since $v, u, w \in X \cap H$

It follows that $X \subset H$. (otherwise by Bézout's Theorem we will get a contradiction)

2) Using Hint : $\deg v_3(\mathbb{P}^1) = 3$

$v_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$[x:y] \longmapsto [x^3 : x^2y : xy^2 : y^3]$

By Lecture $v_3(\mathbb{P}^1)$ is a projective alg. set in $\mathbb{P}^3$ and

$v_3(\mathbb{P}^1) \simeq \mathbb{P}^1$ so it is an irreducible cubic ($\dim v_3(\mathbb{P}^1) = 1$ and $\deg v_3(\mathbb{P}^1) = 3$)

Assume $v_3(\mathbb{P}^1)$ is contained in some hyperplane

$H = V(a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3) \subseteq \mathbb{P}^3$ (with not all $a_i = 0$)

Then $\forall [x:y] \in \mathbb{P}^1$ we have $a_0X^3 + a_1X^2Y + a_2XY^2 + a_3Y^3 = 0$

Hence $V^P(a_0X^3 + a_1X^2Y + a_2XY^2 + a_3Y^3) = \mathbb{P}^1$

but $I(\mathbb{P}^1) = (0) \Rightarrow a_0X^3 + a_1X^2Y + a_2XY^2 + a_3Y^3 = 0$ as a polynomial

$\Rightarrow a_0 = a_1 = a_2 = a_3 = 0$ $\lightning$

Problem 5

1)

Let $w = (x,y,z) \in W = V(F,G)$

Write $\quad \mathrm{Jac}(w) = \begin{bmatrix} 2x & 2y & -2z \\ 2x & 2y-1 & 2z \end{bmatrix}$

$\dim T_{W,w} = 3 - \mathrm{rk}\,\mathrm{Jac}(w)$

$w \in W^{sing} \iff \dim T_{W,w} > \dim_w X \underset{\uparrow}{=} \dim X = 1 \iff \mathrm{rk}\,\mathrm{Jac}(w) < 2$

X irreducible

$\iff$ all 2×2 minors $=0$

In particular $\quad \det \begin{bmatrix} 2x & 2y \\ 2x & 2y-1 \end{bmatrix} = -2x = 0 \Rightarrow x = 0$

$$\det \begin{bmatrix} 2y & -2z \\ 2y-1 & 2z \end{bmatrix} = 4yz + 4yz - 2z = 2z(4y-1) = 0$$

$$\Rightarrow z = 0 \text{ or } y = \tfrac{1}{4}$$

If $x=0, \cancel{y}z=0$ and since $w \in W$, $x^2+y^2-z^2=0$, then $y=0$.

$(0,0,0) \in W^{sing}$ $\quad$ (~~all~~ $\mathrm{rk}\,\mathrm{Jac}(w) = 1$, for $w=(0,0,0)$)

If $x=0, y=\frac{1}{4}$, then (since $w\in W$ and $x=0$): $\quad y^2 = z^2$ and $y^2 - y + z^2 = 0$

$\Rightarrow y = 2y^2 \Rightarrow y=0$ or $y = \frac{1}{2}$

but $y = \frac{1}{4}$, hence the case $x=0, y=\frac{1}{4}$ does not give any singular point on W

2)

X, Y irreducible hypersurfaces in $\mathbb{A}^n \Rightarrow$ $I(X) = (f), I(Y) = (g)$ where $f, g \in K[X_1, \dots, X_n]$ irreducible.

Since $X \neq Y$, $f \neq \alpha g$ $\forall \alpha \in K$.

X and Y smooth $\iff d_a f \neq 0$ and $d_b g \neq 0$ $\forall b \in Y, a \in X$.

$X \cup Y = V(fg)$, hence $I_{X\cup Y} = \sqrt{(fg)} = (fg)$

Let $b \in X \cup Y$ and $b \in X \cap Y$

Then $d_b(fg) = f(b)\, d_b g + g(b)\, d_b f = 0 \Rightarrow b \in (X\cup Y)^{sing}$.

both $=0$, since $b \in X \cap Y$

Assume $b \in X\cup Y$ but $b \notin X \cap Y$

Then $d_b(fg) = f(b)\, d_b g + g(b)\, d_b f \neq 0$ (since $d_b g \neq 0$ for $b \in Y$ and $d_b f \neq 0$ for $b \in X$)

exactly one of them zero $\quad$ Hence $b \notin (X \cup Y)^{sing}$